

LOCAL DERIVATIONS ON SUBALGEBRAS OF τ -MEASURABLE OPERATORS WITH RESPECT TO SEMI-FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. This paper is devoted to local derivations on subalgebras on the algebra $S(M, \tau)$ of all τ -measurable operators affiliated with a von Neumann algebra M without abelian summands and with a faithful normal semi-finite trace τ . We prove that if \mathcal{A} is a solid $*$ -subalgebra in $S(M, \tau)$ such that $p \in \mathcal{A}$ for all projection $p \in M$ with finite trace, then every local derivation on the algebra \mathcal{A} is a derivation. This result is new even in the case standard subalgebras on the algebra $B(H)$ of all bounded linear operators on a Hilbert space H . We also apply our main theorem to the algebra $S_0(M, \tau)$ of all τ -compact operators affiliated with a semi-finite von Neumann algebra M and with a faithful normal semi-finite trace τ .

1. INTRODUCTION

Given an algebra \mathcal{A} , a linear operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule). Each element $a \in \mathcal{A}$ implements a derivation D_a on \mathcal{A} defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be *inner derivations*. If the element a , implementing the derivation D_a , belongs to a larger algebra \mathcal{B} containing \mathcal{A} , then D_a is called a *spatial derivation* on \mathcal{A} . A well known direction in the study of the local action of derivations is the local derivation problem. Recall that a linear map Δ of \mathcal{A} is called a *local derivation* if for each $x \in \mathcal{A}$, there exists a derivation $D : \mathcal{A} \rightarrow \mathcal{A}$, depending on x , such that $\Delta(x) = D(x)$. This notion was introduced in 1990 independently by Kadison [19] and Larson and Sourour [20]. In [19] it was proved that every norm continuous local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. In [20] the same result was obtained for the algebra of all bounded linear operators acting on a Banach space.

In the last decade the structure of derivations and local derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a von Neumann algebra M and on its various subalgebras have been investigated by many authors (see [2–6, 8–16, 18]). In [4] local derivations have been investigated on the algebra $S(M)$ of all measurable operators with respect a von Neumann algebra M . In particular, it was proved that, for finite type I von Neumann algebras without abelian direct summands, every local derivation on $S(M)$ is a derivation. Moreover, in the case of abelian von Neumann algebra M necessary and sufficient conditions have been obtained for the algebra $S(M)$ to admit local derivations which are not derivations. In [18] local derivations have been investigated on the algebra $S(M)$ for an arbitrary von Neumann algebra M and it was proved that for a von Neumann algebras without abelian direct summands every local derivation on $S(M)$ is a derivation. It should be noted that the proofs of the main result in the paper [18] are essentially based on the fact that the von Neumann algebra M is a subalgebra in the considered algebras. Local and 2-local maps have been studied on different operator algebras by many authors [4, 7, 10, 11, 17–20].

The present paper is devoted to local derivations on subalgebras of algebra $S(M, \tau)$ of all τ -measurable operators affiliated with a von Neumann algebra M with a faithful normal semi-finite trace τ . Since in general case we do not assumed that these subalgebras contain the von Neumann algebra M , one cannot directly apply the methods of the papers [18] in this setting. Moreover, in our setting description of local derivations is an open problem. Therefore the aim of this paper to solve such a problem.

In Section 2 we give preliminaries from the theory of τ -measurable operators affiliated with a von Neumann algebra M .

In section 3 we consider a von Neumann algebra M without abelian summands and with a faithful normal semi-finite trace τ . We prove that if \mathcal{A} is a solid $*$ -subalgebra in $S(M, \tau)$ such that $p \in \mathcal{A}$ for all projection $p \in M$ with a finite trace, then every local derivation Δ on the algebra \mathcal{A} is a derivation.

In section 4 we apply the main theorem of the previous section to the Arens algebra and the algebra of all τ -compact operators affiliated with a semi-finite von Neumann algebra M and with a faithful normal semi-finite trace τ .

2. ALGEBRAS OF τ -MEASURABLE OPERATORS

Let $B(H)$ be the $*$ -algebra of all bounded linear operators on a Hilbert space H , and let $\mathbf{1}$ be the identity operator on H . Consider a von Neumann algebra $M \subset B(H)$ with the operator norm $\|\cdot\|$ and with a faithful normal semi-finite trace τ . Denote by $P(M) = \{p \in M : p = p^2 = p^*\}$ the lattice of all projections in M and $P_\tau(M) = \{p \in P(M) : \tau(p) < +\infty\}$.

A linear subspace \mathcal{D} in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M .

A linear operator $x : \mathcal{D}(x) \rightarrow H$, where the domain $\mathcal{D}(x)$ of x is a linear subspace of H , is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary $u \in M'$.

A linear subspace \mathcal{D} in H is said to be *strongly dense* in H with respect to the von Neumann algebra M , if

1. $\mathcal{D}\eta M$;
2. there exists a sequence of projections $\{p_n\}_{n=1}^\infty$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H .

Denote by $S(M)$ the set of all linear operators on H , measurable with respect to the von Neumann algebra M . If $x \in S(M)$, $\lambda \in \mathbb{C}$, where \mathbb{C} is the field of complex numbers, then $\lambda x \in S(M)$ and the operator x^* , adjoint to x , is also measurable with respect to M (see [23]). Moreover, if $x, y \in S(M)$, then the operators $x + y$ and xy are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators x and y , and are denoted by $x \dot{+} y$ and $x \dot{*} y$. It was shown in [23] that $x \dot{+} y$ and $x \dot{*} y$ belong to $S(M)$ and these algebraic operations make $S(M)$ a $*$ -algebra with the identity $\mathbf{1}$ over the field \mathbb{C} . Here, M is a $*$ -subalgebra of $S(M)$. In what

follows, the strong sum and the strong product of operators x and y will be denoted in the same way as the usual operations, by $x + y$ and xy .

It is clear that if the von Neumann algebra M is finite then every linear operator affiliated with M is measurable and, in particular, a self-adjoint operator is measurable with respect to M if and only if all its spectral projections belong to M .

Let τ be a faithful normal semi-finite trace on M . We recall that a closed linear operator x is said to be τ -measurable with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is τ -dense in H , i.e. $\mathcal{D}(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^\perp) < \varepsilon$. Denote by $S(M, \tau)$ the set of all τ -measurable operators affiliated with M .

Note that if the trace τ is finite then $S(M, \tau) = S(M)$.

Consider the topology t_τ of convergence in measure or *measure topology* on $S(M, \tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists e \in P(M), \tau(e^\perp) < \delta, xe \in M, \|xe\| < \varepsilon\},$$

where ε, δ are positive numbers.

It is well-known [22] that $S(M, \tau)$ equipped with the measure topology is a complete metrizable topological $*$ -algebra.

3. LOCAL DERIVATIONS ON $*$ -SUBALGEBRAS OF τ -MEASURABLE OPERATORS

Recall that $*$ -subalgebra $\mathcal{A} \subset S(M, \tau)$ is said to be solid, if $x \in S(M, \tau)$ and $y \in \mathcal{A}$ satisfy $|x| \leq |y|$, then $x \in \mathcal{A}$.

The main result of this section is the following

Theorem 3.1. *Let M be a semi-finite von Neumann algebra without abelian direct summands and let τ be a faithful normal semi-finite trace on M . Suppose that \mathcal{A} is a solid $*$ -subalgebra in $S(M, \tau)$ such that $p \in \mathcal{A}$ for all $p \in P_\tau(M)$. Then every local derivation Δ on the algebra \mathcal{A} is a derivation.*

For the proof of this theorem we need several lemmata.

Let M be a von Neumann algebra and let $x \in S(M, \tau)$. The projection

$$r(x) = \inf\{p \in P(M) : xp = x\}$$

is called the *right support* of the element x , and the projection

$$l(x) = \inf\{p \in P(M) : px = x\}$$

is called its *left support*. The projection $s(x) = r(x) \vee l(x)$ is called the support of the element x . For $*$ -subalgebra $\mathcal{A} \subset S(M, \tau)$ denote

$$\mathcal{F}_\tau(\mathcal{A}) = \{x \in \mathcal{A} : s(x) \in P_\tau(M)\}.$$

The following properties of $\mathcal{F}_\tau(\mathcal{A})$ directly follow from the definition.

Lemma 3.2. *Let \mathcal{A} be a $*$ -subalgebra in $S(M, \tau)$. Then the following assertions are equivalent:*

- (1) $x \in \mathcal{F}_\tau(\mathcal{A})$;
- (2) $\exists p \in P_\tau(M)$ such that $px = x$;
- (3) $\exists p \in P_\tau(M)$ such that $xp = x$;
- (4) $\exists p \in P_\tau(M)$ such that $pxp = x$.

From this lemma we immediately get

Corollary 3.3. $\mathcal{F}_\tau(\mathcal{A})$ is an ideal in \mathcal{A} .

Lemma 3.4. Let Δ be a local derivation on \mathcal{A} . Then $\Delta(\mathcal{F}_\tau(\mathcal{A})) \subset \mathcal{F}_\tau(\mathcal{A})$.

Proof. Take any $x \in \mathcal{F}_\tau(\mathcal{A})$. Then due to locality of Δ one can find a derivation D on \mathcal{A} such that $\Delta(x) = D(x)$. It is clear that

$$\begin{aligned} l(D(x)s(x)) &\preceq s(x), \\ r(xD(s(x))) &\preceq s(x), \end{aligned}$$

where $p \preceq q$ means that p is equivalent to a subprojection of the projection q . Since

$$D(x) = D(xs(x)) = D(x)s(x) + xD(s(x))$$

we have

$$\begin{aligned} \tau(s(D(x))) &= \tau(s(D(x)s(x) + xD(s(x)))) \leq \\ &\leq \tau(s(x) \vee l(D(x)s(x)) \vee r(xD(s(x)))) \leq \\ &\leq \tau(s(x)) + \tau(s(x)) + \tau(s(x)) = 3\tau(s(x)), \end{aligned}$$

i.e.

$$\tau(s(D(x))) \leq 3\tau(s(x)).$$

Thus $D(x) \in \mathcal{F}_\tau(\mathcal{A})$, so $\Delta(x) \in \mathcal{F}_\tau(\mathcal{A})$.

Therefore, Δ maps $\mathcal{F}_\tau(\mathcal{A})$ into itself. □

Let $p \in \mathcal{A}$ be a projection. One can see that the mapping

$$(3.1) \quad D^{(p)} : x \rightarrow pD(x)p, \quad x \in p\mathcal{A}p$$

is a derivation on $p\mathcal{A}p$. Now let Δ be a local derivation on \mathcal{A} . Take an element $x \in \mathcal{A}$ and a derivation D on \mathcal{A} such that $\Delta(pxp) = D(pxp)$. Then

$$p\Delta(pxp)p = pD(pxp)p = D^{(p)}(pxp).$$

This means that the mapping $\Delta^{(p)}$ defined similar to (3.1) is a local derivation on $p\mathcal{A}p$.

Lemma 3.5. If Δ is a local derivation on \mathcal{A} , then the restriction $\Delta|_{\mathcal{F}_\tau(\mathcal{A})}$ is a derivation.

Proof. Let $x, y \in \mathcal{F}_\tau(\mathcal{A})$. Denote

$$p = s(x) \vee s(y) \vee s(xy) \vee s(\Delta(x)) \vee s(\Delta(y)) \vee s(\Delta(xy)).$$

Since $\mathcal{F}_\tau(\mathcal{A})$ is an ideal in \mathcal{A} and Δ maps $\mathcal{F}_\tau(\mathcal{A})$ into itself, we obtain that the projection $p \in P_\tau(M)$. Consider the local derivation $\Delta^{(p)}$ on $p\mathcal{A}p$. Since $p \in \mathcal{A}$ and \mathcal{A} is a solid $*$ -subalgebra in $S(M, \tau)$ we get $pMp \subseteq \mathcal{A}$. Furthermore, \mathcal{A} has no abelian direct summands, and therefore [18, Theorem 1] implies that $\Delta^{(p)}$ is a derivation. Taking into account that $x, y \in p\mathcal{A}p$ we obtain

$$\Delta^{(p)}(xy) = \Delta^{(p)}(x)y + x\Delta^{(p)}(y).$$

By construction of the projection p we have

$$\begin{aligned} \Delta(xy) &= \Delta^{(p)}(xy) = \Delta^{(p)}(x)y + x\Delta^{(p)}(y) = \\ &= \Delta(x)y + x\Delta(y). \end{aligned}$$

This means that Δ is a derivation on $\mathcal{F}_\tau(\mathcal{A})$. This completes the proof. \square

Remark 3.6. Let $y \in \mathcal{A}$ and $yp = 0$ for all $p \in P_\tau(M)$. Since the map $x \mapsto yxy^*$ is positive and monotone continuous, taking $p \uparrow \mathbf{1}$ in $ypy^* = 0$, we obtain that $yy^* = 0$. Therefore $y = 0$.

Proof of Theorem 3.1. We shall show that

$$\Delta(xy) = \Delta(x)y + x\Delta(y)$$

for all $x, y \in \mathcal{A}$. We consider the following two cases.

CASE 1. Let $x \in \mathcal{F}_\tau(\mathcal{A})$ and $y \in \mathcal{A}$. Since $\mathcal{F}_\tau(\mathcal{A})$ is an ideal in \mathcal{A} and Δ maps $\mathcal{F}_\tau(\mathcal{A})$ into itself, we obtain that the projection

$$p = s(xy) \vee s(\Delta(xy)) \vee s(\Delta(x)y) \vee s(x\Delta(y))$$

has a finite trace. Taking into account the equalities

$$xyp = xy, \Delta(xy)p = \Delta(xy)$$

and Lemma 3.5 we obtain

$$\begin{aligned} \Delta(xy) &= \Delta(xyp) = \Delta(xy)p + xy\Delta(p) = \\ &= \Delta(xy) + xy\Delta(p), \end{aligned}$$

i.e. $xy\Delta(p) = 0$.

Further

$$\begin{aligned} x\Delta(y)p &= x\Delta(ypp) - xyp\Delta(p) = \\ &= x\Delta(y)p - xy\Delta(p) = x\Delta(y)p, \end{aligned}$$

i.e.

$$(3.2) \quad x\Delta(y)p = x\Delta(y)p.$$

Now taking into account (3.2), the equalities

$$xy(\mathbf{1} - p) = 0, x\Delta(y)p = x\Delta(y)$$

and the linearity of Δ we have

$$\begin{aligned} x\Delta(y) &= x\Delta(y)p = x\Delta(y(p + y(\mathbf{1} - p)))p = \\ &= x\Delta(y)p + x\Delta(y(\mathbf{1} - p))p = x\Delta(y)p + xD(y(\mathbf{1} - p))p = \\ &= x\Delta(y)p + xD(y(\mathbf{1} - p)p) - xy(\mathbf{1} - p)D(p) = x\Delta(y)p, \end{aligned}$$

where D is a derivation on \mathcal{A} such that $\Delta(y(\mathbf{1} - p)) = D(y(\mathbf{1} - p))$. Consequently

$$x\Delta(y)p = x\Delta(y).$$

Finally we obtain that

$$\begin{aligned} \Delta(xy) &= \Delta(xyp) = \Delta(x)yp + x\Delta(y)p = \\ &= \Delta(x)y + x\Delta(y). \end{aligned}$$

Similar as above we can check the case $x \in \mathcal{A}$ and $y \in \mathcal{F}_\tau(\mathcal{A})$.

CASE 2. Let $x, y \in \mathcal{A}$ be arbitrary elements. Take an arbitrary $q \in P_\tau(M)$. By the case 1 we have

$$\Delta(y)q = \Delta(yq) - y\Delta(q).$$

Taking into account this equality and the case 1 we obtain

$$\begin{aligned} \Delta(xy)q &= \Delta(xyq) - xy\Delta(q) = \\ &= \Delta(x)yq + x\Delta(yq) - xy\Delta(q) = \\ &= \Delta(x)yq + x[\Delta(yq) - y\Delta(q)] = \\ &= \Delta(x)yq + x\Delta(y)q, \end{aligned}$$

i.e.

$$\Delta(xy)q = [\Delta(x)y + x\Delta(y)]q$$

for all $q \in P_\tau(M)$. Taking into account Remark 3.6 we obtain

$$\Delta(xy) = \Delta(x)y + x\Delta(y).$$

The proof is complete. □

We stress that Theorem 3.1 is new even in the case of type I_∞ von Neumann factors.

For a Hilbert space H by $\mathcal{F}(H)$ we denote the algebra of all finite rank operators in $B(H)$. Recall that a standard operator algebra is any subalgebra of $B(H)$ which contains $\mathcal{F}(H)$.

Theorem 3.1 implies the following result.

Corollary 3.7. *Let H be a Hilbert space and let \mathcal{U} be a standard algebra in $B(H)$. Then any local derivation $\Delta : \mathcal{U} \rightarrow \mathcal{U}$ is a spatial derivation and implemented by an element $a \in B(H)$.*

Remark 3.8. *A similar result for local derivations on $B(X)$, where X is a Banach space, has been obtained in [17, Corollary 3.7] under the additional assumption of continuity of the map with respect to the weak operator topology.*

Remark 3.9. *We note that if one replaces $S(M, \tau)$ with $S(M)$ all the results will remain true. In this case $\mathcal{F}_\tau(\mathcal{A})$ is replaced by the set of finite projections of \mathcal{A} and instead of τ is used the dimension function.*

4. LOCAL DERIVATIONS ON ALGEBRA τ -COMPACT OPERATORS AND ARENS ALGEBRAS

In this section we shall consider a local derivations on algebras τ -compact operators and on Arens algebras, respectively.

4.1. algebra of τ -compact operators. In this subsection we shall consider an algebra of τ -compact operators.

In the algebra $S(M, \tau)$ consider the subset $S_0(M, \tau)$ of all operators x such that given any $\varepsilon > 0$ there is a projection $p \in P(M)$ with $\tau(p^\perp) < \infty$, $xp \in M$ and $\|xp\| < \varepsilon$. The elements of $S_0(M, \tau)$ is called τ -compact operators with respect to M and τ . It is known [21] that $S_0(M, \tau)$ is a solid $*$ -subalgebra in $S(M, \tau)$ and a bimodule over M , i.e. $ax, xa \in S_0(M, \tau)$ for all $x \in S_0(M, \tau)$ and $a \in M$. Note that if $M = B(H)$ and $\tau = tr$, where tr is the canonical trace on $B(H)$, then $S_0(M, \tau) = K(H)$, where $K(H)$ is the ideal of compact operators from $B(H)$.

The following properties of the algebra $S_0(M, \tau)$ are known (see [24]):

Let M be a von Neumann algebra with a faithful normal semi-finite trace τ . Then

- 1) $S(M, \tau) = M + S_0(M, \tau)$;
- 2) $S_0(M, \tau)$ is an ideal in $S(M, \tau)$.

Note that if the trace τ is finite then

$$S_0(M, \tau) = S(M, \tau) = S(M).$$

It is well-known [24] $S_0(M, \tau)$ equipped with the measure topology is a complete metrizable topological $*$ -algebra.

It is clear that $p \in S_0(M, \tau)$ for all $p \in P_\tau(M)$.

Theorem 3.1 implies the following result.

Theorem 4.1. *Let M be a semi-finite von Neumann algebra without abelian direct summands and let τ be a faithful normal semi-finite trace on M . Then every local derivation Δ on the algebra $S_0(M, \tau)$ is a derivation.*

Remark 4.2. *If M is an abelian von Neumann algebra with a faithful normal semi-finite trace τ such that the lattice $P(M)$ of projections in M is not atomic, then the algebra $S_0(M, \tau)$ admits a local derivation which is not a derivation (see [4, Theorem 3.2]).*

In [9, Theorem 4.9] it was proved that in the case when M is a properly infinite von Neumann algebra with a faithful normal semi-finite trace τ , then any derivation D on $S_0(M, \tau)$ is a spatial derivation and implemented by an element $a \in S(M, \tau)$. Therefore Theorem 4.1 implies that

Theorem 4.3. *If M is a properly infinite von Neumann algebra with a faithful normal semi-finite trace τ , then any local derivation $\Delta : S_0(M, \tau) \rightarrow S_0(M, \tau)$ is a spatial derivation and implemented by an element $a \in S(M, \tau)$.*

4.2. Arens algebras. Now we are going to consider Arens algebras associated with a von Neumann algebra and a semi-finite faithful normal trace.

Let M be a von Neumann algebra with a faithful normal semi-finite trace τ .

Take $x \in S(M, \tau)$, $x \geq 0$ and let $x = \int_0^\infty \lambda de_\lambda$ be its spectral resolution. Denote $\tau(x) = \sup_{n \geq 1} \int_0^n \lambda d\tau(e_\lambda)$.

Given $p \geq 1$ put $L^p(M, \tau) = \{x \in S(M, \tau) : \tau(|x|^p) < \infty\}$. It is known [21] that $L^p(M, \tau)$ is a Banach space with respect to the norm

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(M, \tau).$$

Consider the intersection

$$L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$

It is proved in [1] that $L^\omega(M, \tau)$ is a locally convex complete metrizable $*$ -algebra with respect to the topology t generated by the family of norms $\{\|\cdot\|_p\}_{p \geq 1}$. The algebra $L^\omega(M, \tau)$ is called a (non commutative) *Arens algebra*.

Note that $L^\omega(M, \tau)$ is a solid $*$ -subalgebra in $S(M, \tau)$ and if τ is a finite trace then $M \subset L^\omega(M, \tau)$.

Further consider the following spaces

$$L_2^\omega(M, \tau) = \bigcap_{p \geq 2} L^p(M, \tau)$$

and

$$M + L_2^\omega(M, \tau) = \{x + y : x \in M, y \in L_2^\omega(M, \tau)\}.$$

It is known [2] that $L_2^\omega(M, \tau)$ and $M + L_2^\omega(M, \tau)$ are a $*$ -algebras and $L^\omega(M, \tau)$ is an ideal in $M + L_2^\omega(M, \tau)$.

Note that if $\tau(\mathbf{1}) < \infty$ then $M + L_2^\omega(M, \tau) = L_2^\omega(M, \tau) = L^\omega(M, \tau)$.

It is known [2, Theorem 3.7] that if M is a von Neumann algebra with a faithful normal semi-finite trace τ then any derivation D on $L^\omega(M, \tau)$ is spatial, moreover it is implemented by an element of $M + L_2^\omega(M, \tau)$, i. e.

$$(4.1) \quad D(x) = ax - xa, \quad x \in L^\omega(M, \tau)$$

for some $a \in M + L_2^\omega(M, \tau)$. In particular, if M is abelian, then any derivation on $L^\omega(M, \tau)$ is zero.

Note that $p \in L^\omega(M, \tau)$ for all $p \in P_\tau(M)$.

We need the following auxiliary result.

Lemma 4.4. *Let M be a semi-finite von Neumann algebra with a faithful normal semi-finite trace τ and with the center $Z(M)$. Then every local derivation Δ on the algebra $L^\omega(M, \tau)$ is necessarily $P(Z(M))$ -homogeneous, i.e.*

$$\Delta(zx) = z\Delta(x)$$

for any central projection $z \in P(Z(M)) = P(M) \cap Z(M)$ and for all $x \in L^\omega(M, \tau)$.

Proof. Take $z \in P(Z(M))$ and $x \in L^\omega(M, \tau)$. For the element zx by the definition of the local derivation Δ there exists a derivation D_a on $L^\omega(M, \tau)$ of the form (4.1) such that $\Delta(zx) = D_a(zx)$. Since the projection z is central, one has that

$$D_a(zx) = [a, zx] = z[a, x] = zD_a(x).$$

Multiplying the equality $\Delta(zx) = D_a(zx)$ by z we obtain

$$z\Delta(zx) = zD_a(zx) = zD_a(x) = D_a(zx) = \Delta(zx),$$

i.e.

$$(\mathbf{1} - z)\Delta(zx) = 0.$$

Replacing z by $\mathbf{1} - z$ one finds

$$z\Delta((\mathbf{1} - z)x) = 0.$$

Therefore by the linearity of Δ we have

$$z\Delta(x) = z\Delta(zx) + z\Delta((\mathbf{1} - z)x) = z\Delta(zx) = \Delta(zx),$$

and thus $z\Delta(x) = \Delta(zx)$. The proof is complete. \square

Theorem 4.5. *Let M be a semi-finite von Neumann algebra with a faithful normal semi-finite trace τ . Then any local derivation Δ on the algebra $L^\omega(M, \tau)$ is a spatial derivation of the form (4.1).*

Proof. Let M be a semi-finite von Neumann algebra. There exist mutually orthogonal central projections z_1, z_2 in M with $z_1 + z_2 = \mathbf{1}$ such that

- z_1M is abelian;
- z_2M has no abelian summands.

By Lemma 4.4 the operator Δ maps $z_i L^\omega(M, \tau) \equiv L^\omega(z_i M, \tau_i)$ into itself for $i = 1, 2$, where τ_i is the restriction of τ on $z_i M$ ($i = 1, 2$). As it was mentioned above $z_1 \Delta$ is zero. By Theorem 3.1 we obtain that $\Delta = z_2 \Delta$ is a derivation. The proof is complete. \square

Note that if the trace τ is finite Theorem 4.5 it was given in [11, Theorem 2.1].

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